2-7 A **group** is a set X together with an operation $*$ defined on that set such that

- $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
- $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$
- For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$

Show that SO(n) with the operation of matrix multiplication is a group.

2-8 Derive Equations (2.6) and (2.7).

2-9 Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2-10 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the world y-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-11 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-12 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.

2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-13 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-14 If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

2-15 Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix R_3^2.

2-16 Derive equations for the roll, pitch, and yaw angles corresponding to the rotation matrix $R = (r_{ij})$.

2-17 Verify Equation (2.43).

2-18 Verify Equation (2.45).

2-19 If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

2-20 Let $k = \frac{1}{\sqrt{3}}[1,1,1]^T$, $\theta = 90°$. Find $R_{k,\theta}$.

2-21 Show by direct calculation that $R_{k,\theta}$ given by Equation (2.43) is equal to R given by Equation (2.47) if θ and k are given by Equations (2.48) and (2.49), respectively.

2-22 Compute the rotation matrix given by the product

$$R_{x,\theta}R_{y,\phi}R_{z,\pi}R_{y,-\phi}R_{x,-\theta}$$

2-23 Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

2-24 Find the rotation matrix corresponding to the Euler angles $\phi = \frac{\pi}{2}$, $\theta = 0$, and $\psi = \frac{\pi}{4}$. What is the direction of the x_1 axis relative to the base frame?

2-25 Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

2-26 Unit magnitude complex numbers $a + ib$ with $a^2 + b^2 = 1$ can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{Atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

2-27 Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

2-28 Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned} -1 &= i^2 = j^2 = k^2, \\ i &= jk = -kj, \\ j &= ki = -ik, \\ k &= ij = -ji \end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = [n_x, n_y, n_z]^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, that is, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

2-29 Using $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$, and the results from Section 2.5.3, determine the rotation matrix R that corresponds to the rotation represented by the quaternion (q_0, q_1, q_2, q_3).

2-30 Determine the quaternion Q that represents the same rotation as given by the rotation matrix R.

2-31 The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $q = [q_1, q_2, q_3]^T$. Show that the product of two quaternions, $Z = XY$ is given by

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

Hint: Perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

2-32 Show that $Q_I = (1, 0, 0, 0)$ is the identity element for unit quaternion multiplication, that is, $QQ_I = Q_I Q = Q$ for any unit quaternion Q.

2-33 The conjugate Q^* of the quaternion Q is defined as

$$Q^* = (q_0, -q_1, -q_2, -q_3)$$

Show that Q^* is the inverse of Q, that is, $Q^* Q = QQ^* = (1, 0, 0, 0)$.

2-34 Let v be a vector whose coordinates are given by $[v_x, v_y, v_z]^T$. If the quaternion Q represents a rotation, show that the new, rotated coordinates of v are given by $Q(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component.

2-35 Let the point p be rigidly attached to the end effector coordinate frame with local coordinates (x, y, z). If Q specifies the orientation of the end effector frame with respect to the base frame, and T is the vector from the base frame to the origin of the end effector frame, show that the coordinates of p with respect to the base frame are given by

$$Q(0, x, y, z)Q^* + T \tag{2.68}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component.

2-36 Verify Equation (2.60).

2-37 Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin o_1 with respect to the original frame in each case?

2-38 Consider the diagram of Figure 2.13. Find the homogeneous transformations H_1^0, H_2^0, H_2^1 representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0, H_2^1$.

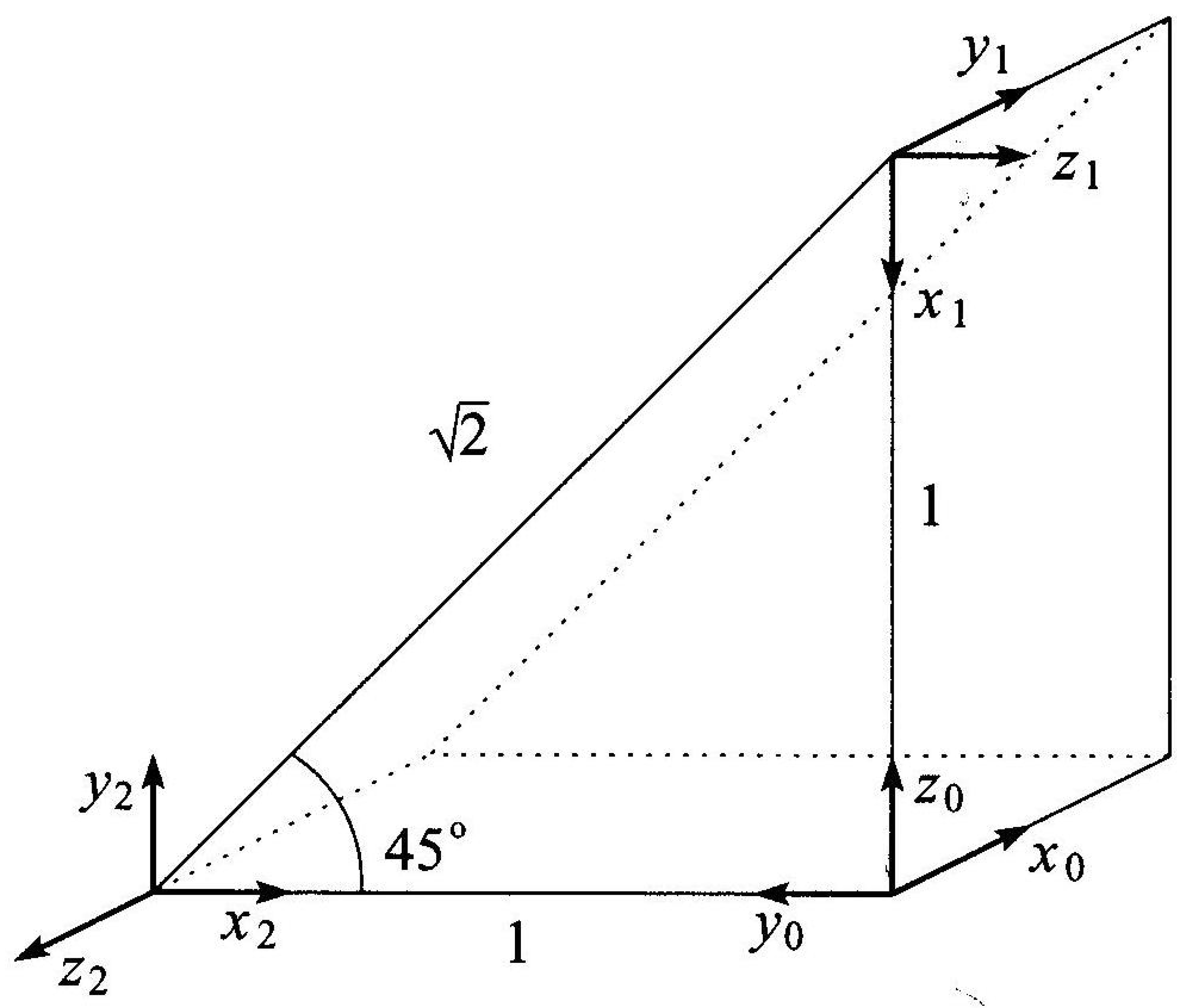

Figure 2.13: Diagram for Problem 2-38.

2-39 Consider the diagram of Figure 2.14. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2 meters above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

2-40 In Problem 2-39, suppose that, after the camera is calibrated, it is rotated 90° about z_3. Recompute the above coordinate transformations.

2-41 If the block on the table is rotated 90° about z_2 and moved so that its center has coordinates $[0, .8, .1]^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.

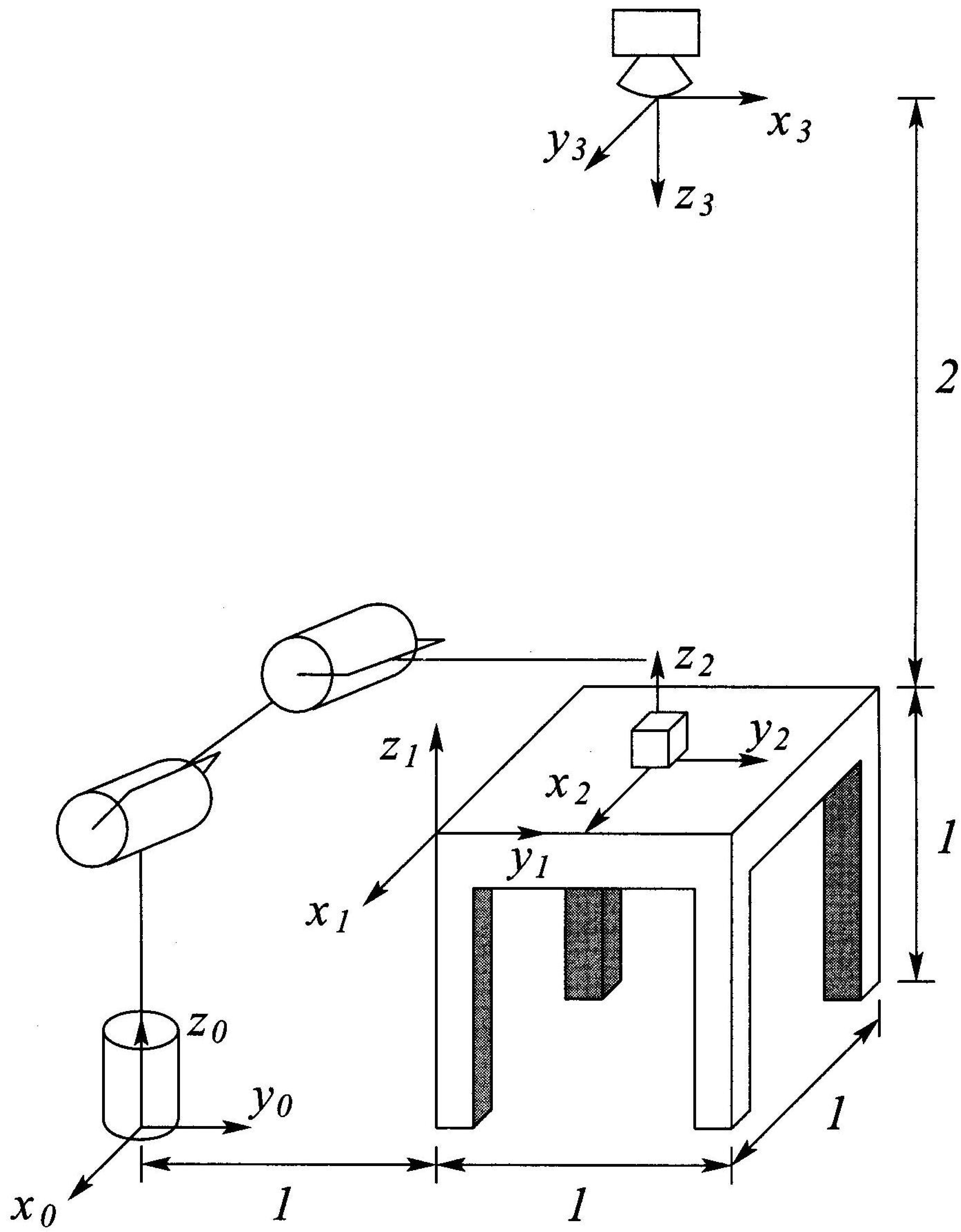

Figure 2.14: Diagram for Problem 2-39.

2-42 Consult an astronomy book to learn the basic details of the Earth's rotation about the sun and about its own axis. Define for the Earth a local coordinate frame whose z-axis is the Earth's axis of rotation. Define $t = 0$ to be the exact moment of the summer solstice, and the global reference frame to be coincident with the Earth's frame at time $t = 0$. Give an expression $R(t)$ for the rotation matrix that represents the instantaneous orientation of the earth at time t. Determine as a function of time the homogeneous transformation that specifies the Earth's frame with respect to the global reference frame.

2-43 In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product